

Analysis of Some Overflow Problems with Queuing

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When calls offered to a primary group of trunks find all of them busy, provisions are often made for these calls to overflow to other groups of trunks. Such traffic overflow systems have been of interest for a long time, but recently overflow systems that allow for some calls to be queued have been of importance. In this paper we analyze a traffic overflow system with queuing. The system consists of two groups, a primary and a secondary. We consider two cases which differ in the treatment of demands waiting in the primary queue. We adopt a novel analytical approach, which considerably reduces the dimensions of the problem and simplifies the calculation of various steady-state quantities of interest. Our theoretical results include expressions for the loss probabilities, the probability of overflow from the primary to the secondary, and the average waiting times in the queues.

I. INTRODUCTION

In this paper, a particular overflow system with queuing is analyzed. The system consists of two groups, a primary and a secondary, with n_k servers and q_k waiting spaces, which receive demands from independent Poisson sources S_k with arrival rates $\lambda_k > 0$, $k = 1$ and 2 , respectively, as depicted in Fig. 1. The service times of the demands are independent and exponentially distributed with mean service rate $\mu > 0$. If all n_2 servers in the secondary are busy when a demand from S_2 arrives, the demand is queued if one of the q_2 waiting spaces is available; otherwise, it is lost (blocked and cleared from the system). Demands waiting in the secondary queue enter service (in some prescribed order) as servers in the secondary become free.

If all n_1 servers in the primary are busy when a demand from S_1 arrives and there is a free server in the secondary and no demands waiting in the secondary queue, the demand is served in the secondary.

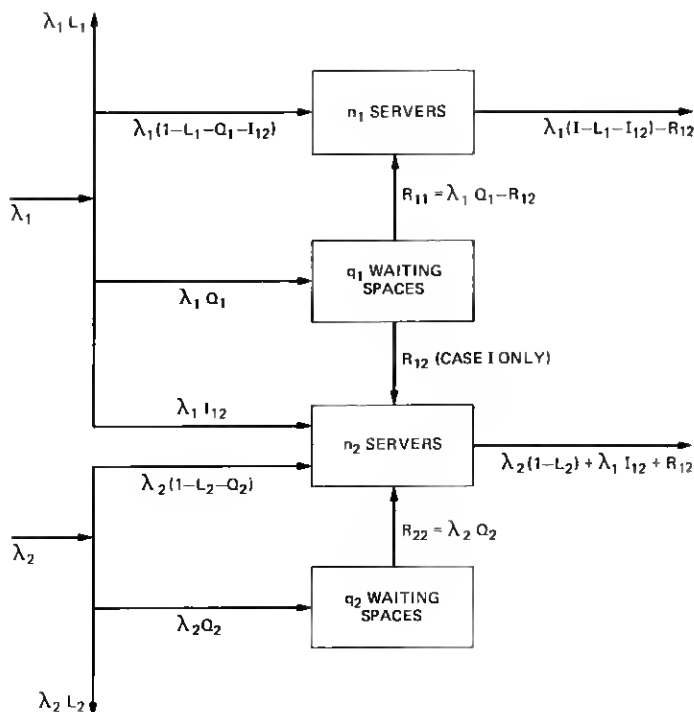


Fig. 1—Mean flow rates for an overflow system with queuing.

If there are no free servers, then the demand is queued in the primary queue, if one of the q_1 waiting spaces is available; otherwise, it is lost. Two different cases are considered for the treatment of demands waiting in the primary queue. In case I, a demand waiting in the primary queue may enter service either in the primary, when a server becomes free, or in the secondary, if a server becomes free and no demands are waiting in the secondary queue. In case II, no overflow is permitted from the primary queue, so that a demand in the primary queue must wait for a server in the primary to become free.

The case $q_1 = 0$ and $q_2 = 0$ corresponds to no queuing. This problem has been analyzed by Brockmeyer¹ in the particular case $\lambda_2 = 0$, corresponding to no secondary source. The case $\lambda_2 \neq 0$ is a special case of two groups of servers with a common overflow group. The first group has n_1 servers and the second none, while the overflow group has n_2 servers. This problem has been analyzed by Kosten,^{2,3} but his analysis differs from ours, as is explained in the next section.

Generally, let p_{ij} denote the steady-state probability that there are i demands in the primary and j demands in the secondary. These probabilities satisfy a set of generalized birth-and-death equations,

which take the form of partial difference equations connecting neighboring states. Rath⁴ has solved these equations numerically in case I, by using a Gauss-Seidel iteration technique. Here we carry out an analysis that reduces the dimensions of the problem, which may be considerable in cases of interest. The basic technique is to separate variables in regions away from certain boundaries of the state space. This leads to various eigenvalue problems for the separation constant. The eigenvalues are roots of polynomial equations. The probabilities p_{ij} are then represented in terms of the corresponding eigenfunctions. The constant coefficients in these representations are determined from the boundary conditions, and the normalization condition that the sum of the probabilities is unity. In general, these constants have to be determined numerically. In case II, additional states are accessible, since in this case demands wait in the primary queue even if a server is free in the secondary. In this additional region, the probabilities p_{ij} are expressed in terms of a fundamental solution of a partial difference equation.

A different procedure for reducing the dimensions of the problem is that of Herzog, Woo, and Chandy.¹³ Their procedure, if applied to the problem of this paper, would involve the recursive determination of a number of solutions of a partial difference equation, in order to express the probabilities p_{ij} in terms of their values on certain boundaries. Except in case II, we have to solve only ordinary difference equations in our approach, because of the separation of variables. In case II, as noted above, it is necessary to obtain only a single solution of a partial difference equation in a certain region.

Various steady-state quantities which are of interest may be expressed in terms of the probabilities p_{ij} . The quantities include the loss (or blocking) probabilities, the probabilities that a demand is queued, the probability of overflow from the primary to the secondary, and the average waiting times in the queues. These quantities may be expressed directly in terms of the constant coefficients which occur in the representations for the probabilities p_{ij} . Thus the steady-state quantities of interest may be calculated directly, once the coefficients have been determined from the boundary and normalization conditions, without having to calculate the probabilities p_{ij} . Here again the reduction in the dimensions of the problem is valuable.

Only the theoretical results are presented in this paper. Numerical results will be presented in a forthcoming paper by Kaufman, Seery and Morrison.⁵ The reduction in the dimensions of the problem by means of the analysis presented in this paper leads to a dense matrix, rather than the much larger sparse matrix for determining the probabilities p_{ij} . Nevertheless, the approach presented here is amenable to numerical calculation for moderately large problems. The largest case

run so far using this approach has been for $n_1 = 40 = n_2$ and $q_1 = 10 = q_2$, for case I. Kaufman⁶ has used a numerical technique involving matrix separability (block diagonalization) for these problems and, in fact, is led to one of the same eigenvalue problems. In addition, Kaufman has obtained numerical solutions by means of successive over-relaxation techniques. For the example mentioned above, our results for the steady-state quantities of interest agree to 12 significant figures, although some of the probabilities p_{ij} , which range over 30 orders of magnitude, agree to fewer significant figures.

An iterative solution of two-dimensional birth-and-death processes has been discussed by Brandwajn.¹⁴ Although his numerical results for some specific problems indicate better execution times than with Gauss-Seidel iteration, the successive over-relaxation technique, with appropriate choice of parameter, would seem to be superior.

The theoretical results are built up in stages, commencing with the analysis of the overflow problem without queuing in Section II. Next, the case in which there is only a secondary queue is investigated in Section III. This analysis pertains to both case I and case II, since the two cases differ only in the treatment of demands queued in the primary. The overflow problem in which there are both a primary and a secondary queue is investigated in Section IV for case I, and the steady-state quantities of interest are calculated in Section V. Another approach for case I, when there is only a primary queue, is presented in Section VI. The analysis of case II, when there is only a primary queue, is given in Section VII. The investigation of case II, when there is both a primary and a secondary queue, is commenced in Section VIII, and the steady-state quantities of interest are calculated in Section IX.

Properties of the eigenfunctions which occur in the representations of the probabilities p_{ij} , and which are related to Poisson-Charlier^{7,8} and Chebyshev⁹ polynomials, are given in Appendix A. Appendix B shows that the corresponding eigenvalues are distinct. Moreover, the eigenvalues, which are roots of polynomial equations, enjoy an interlacing property which facilitates their numerical evaluation.

Another approach to the solution of the overflow problem, which is useful when q_1 is large, or even infinite, as is the analysis in Section VI for case I with $q_2 = 0$, will be presented in a separate paper. Also, an alternate approach, which is useful when q_2 is large, or even infinite, is presented in another paper.¹⁵

II. OVERFLOW WITHOUT QUEUING

We begin by considering an overflow system without queuing. The system consists of two groups, a primary and a secondary, with $n_k \geq 1$ servers, which receive demands from independent Poisson sources

S_k with arrival rates $\lambda_k > 0$, $k = 1$ and 2 , respectively. The service times of the demands are independent and exponentially distributed with mean service rate $\mu > 0$. If all n_2 servers in the secondary are busy when a demand from S_2 arrives, the demand is lost (blocked and cleared from the system). If all n_1 servers in the primary are busy when a demand from S_1 arrives, then the demand is served in the secondary, if one of the n_2 servers is free; otherwise, the demand is lost.

The overflow system corresponding to the case of no secondary source (or $\lambda_2 = 0$) has been analyzed by Brockmeyer.¹ Also, the system described above is a special case of two groups of servers with a common overflow group. The first group has n_1 servers and the second none, while the overflow group has n_2 servers. This problem has been analyzed by Kosten,^{2,3} but his analysis differs from ours, as will be explained.

Let p_{ij} denote the steady-state probability that there are i demands in the primary and j demands in the secondary. These probabilities satisfy a set of generalized birth-and-death equations,¹⁰ which may be derived in a straightforward manner. We define the traffic intensities

$$a_1 = \lambda_1/\mu, \quad a_2 = \lambda_2/\mu, \quad (1)$$

and let δ_{ij} denote the Kronecker delta; $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ if $i \neq j$. Then it is found that

$$\begin{aligned} & [a_1(1 - \delta_{in_1}\delta_{jn_2}) + a_2(1 - \delta_{jn_2}) + i + j]p_{ij} \\ & = a_1(1 - \delta_{i0})p_{i-1,j} + (1 - \delta_{j0})(a_1\delta_{in_1} + a_2)p_{i,j-1} \\ & \quad + (1 - \delta_{in_1})(i + 1)p_{i+1,j} + (1 - \delta_{jn_2})(j + 1)p_{i,j+1}, \quad (2) \end{aligned}$$

for $0 \leq i \leq n_1$, $0 \leq j \leq n_2$. The normalization condition is

$$\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} p_{ij} = 1. \quad (3)$$

It is known from the theory of finite Markov chains¹¹ that there is a unique solution to the system (2) and (3), and $p_{ij} > 0$. Once this solution has been obtained, the steady-state quantities of interest may be calculated. In particular, if L_k is the probability that a demand arriving from S_k is lost, then

$$L_1 = p_{n_1, n_2}, \quad L_2 = \sum_{i=0}^{n_1} p_{i, n_2}. \quad (4)$$

Note that a demand from the primary source is lost only if all the servers in both the primary and the secondary are busy. Such a demand overflows to the secondary if all n_1 servers in the primary are busy, but there is a server free in the secondary, so that the probability

of overflow is

$$I_{12} = \sum_{j=0}^{n_2-1} p_{n_1,j}. \quad (5)$$

We now proceed to solve the system (2). If $i \neq n_1$, the variables in (2) may be separated, and there are solutions of the form $\alpha_i \beta_j$ where

$$(a_1 + i + \rho)\alpha_i = a_1(1 - \delta_{i0})\alpha_{i-1} + (i + 1)\alpha_{i+1}, \quad (6)$$

for $0 \leq i \leq n_1 - 1$, and

$$[a_2(1 - \delta_{jn_2}) + j - \rho]\beta_j = a_2(1 - \delta_{j0})\beta_{j-1} + (1 - \delta_{jn_2})(j + 1)\beta_{j+1}, \quad (7)$$

for $0 \leq j \leq n_2$, and ρ is a separation constant. The solution of (6) may be expressed in terms of Poisson-Charlier polynomials.^{7,8} We here denote the solution of (6) for which $\alpha_0 = 1$ by $s_i(\rho, a_1)$. The properties of $s_i(\rho, a)$ which we will need are given in Appendix A.

For $0 \leq j \leq n_2 - 1$, a solution of (7) is $\beta_j = s_j(-\rho, a_2)$, $0 \leq j \leq n_2$. The boundary condition at $j = n_2$ implies that

$$(n_2 - \rho)s_{n_2}(-\rho, a_2) = a_2 s_{n_2-1}(-\rho, a_2). \quad (8)$$

With the help of (169) and (170) in Appendix A, this equation may be written in the form

$$\rho s_{n_2}(1 - \rho, a_2) = 0. \quad (9)$$

But $s_{n_2}(1 - \rho, a_2)$ is a polynomial in ρ of degree n_2 . It is shown in Appendix B that its zeros are positive and distinct, and we denote them by ρ_m , $m = 1, \dots, n_2$. Then, with $\rho_0 = 0$, we have

$$\rho_m s_{n_2}(1 - \rho_m, a_2) = 0, \quad m = 0, \dots, n_2. \quad (10)$$

It follows that we may represent the probabilities p_{ij} in the form

$$p_{ij} = \sum_{m=0}^{n_2} c_m s_i(\rho_m, a_1) s_j(-\rho_m, a_2), \quad 0 \leq i \leq n_1, \quad 0 \leq j \leq n_2, \quad (11)$$

where the constants c_m are to be determined. It remains to satisfy the boundary conditions corresponding to $i = n_1$ in (2), as well as the normalization condition (3). Hence, for $i = n_1$ and $j \neq n_2$, we must have

$$\begin{aligned} (a_1 + a_2 + n_1 + j)p_{n_1,j} \\ = a_1 p_{n_1-1,j} + (a_1 + a_2)(1 - \delta_{j0})p_{n_1,j-1} + (j + 1)p_{n_1,j+1}, \end{aligned} \quad (12)$$

for $0 \leq j \leq n_2 - 1$. If we substitute (11) into (12), it is found, after reduction with the help of the recurrence relations in Appendix A, that

$$\sum_{m=0}^{n_2} c_m [\rho_m s_{n_1} (1 + \rho_m, a_1) s_j (-\rho_m, a_2) + a_1 s_{n_1} (\rho_m, a_1) s_j (-1 - \rho_m, a_2)] = 0, \quad (13)$$

for $0 \leq j \leq n_2 - 1$.

The boundary condition corresponding to $i = n_1$ and $j = n_2$ in (2), namely,

$$(n_1 + n_2) p_{n_1, n_2} = a_1 p_{n_1-1, n_2} + (a_1 + a_2) p_{n_1, n_2-1}, \quad (14)$$

implies, after reduction and the use of (10), that

$$\sum_{m=0}^{n_2} c_m [\rho_m s_{n_1} (1 + \rho_m, a_1) s_{n_2-1} (1 - \rho_m, a_2) + a_1 s_{n_1} (\rho_m, a_1) s_{n_2-1} (-\rho_m, a_2)] = 0. \quad (15)$$

As is to be expected, this condition is redundant, as may be verified by summing (13) from $j = 0$ to $n_2 - 1$, and using (171) in Appendix A. Thus the coefficients c_m are determined by (13) only to within a multiplicative constant, which is determined from the normalization condition (3). From (3), (10), (11), and (171), it follows that

$$c_0 s_{n_1} (1, a_1) s_{n_2} (1, a_2) = 1. \quad (16)$$

In general, the constants c_m have to be calculated numerically. In the limiting case $a_2 = 0$, for which $\rho_m = m$, from (10) and (167), we were able to obtain an analytical expression for c_m , and to reproduce the results of Brockmeyer,¹ but we omit the details.

We now comment on the approach adopted by Kosten.^{2,3} Whereas we obtained an eigenvalue problem by satisfying the boundary conditions at $j = n_2$, but not at $i = n_1$, Kosten obtained an eigenvalue problem by satisfying the boundary conditions at $i = n_1$, but not at $j = n_2$. He obtains the eigenvalue equation $\gamma s_{n_1} (1 + \gamma, a_1) = 0$, with roots $\gamma_0 = 0$ and $\gamma_r, r = 1, \dots, n_1$. However, since the variables cannot be separated in (2) in the region $0 \leq i \leq n_1, 0 \leq j \leq n_2 - 1$, the corresponding eigenfunctions, which occur in the representation of p_{ij} , contain two terms for $r = 1, \dots, n_1$, rather than one as in (11). Moreover, the eigenfunction corresponding to $\gamma_0 = 0$, which corresponds to the solution for $n_2 = \infty$, is an infinite sum. Hence it would seem that Kosten's approach is preferable if n_2 is significantly larger than n_1 , but that the representation (11) is more suitable if $n_2 \leq n_1$.

Once the constants c_m have been determined, the probabilities p_{ij} may be calculated from (11). We note, in particular, that, with the help

of (10) and (16), and (167) and (171) in Appendix A, it follows that

$$\sum_{j=0}^{n_2} p_{ij} = \frac{s_i(0, a_1)}{s_{n_1}(1, a_1)} = \frac{a_1^i / i!}{\sum_{r=0}^{n_1} a_1^r / r!}. \quad (17)$$

This is the well-known result for the steady-state probability that there are i demands in the primary. From (4), (5), and (11), with the help of (171), we obtain

$$L_2 = \sum_{m=0}^{n_2} c_m s_{n_1}(1 + \rho_m, a_1) s_{n_2}(-\rho_m, a_2) \quad (18)$$

and

$$I_{12} = \sum_{m=0}^{n_2} c_m s_{n_1}(\rho_m, a_1) s_{n_2-1}(1 - \rho_m, a_2). \quad (19)$$

Also, from (4), (5), and (17), we have

$$L_1 + I_{12} = \sum_{j=0}^{n_2} p_{n_1, j} = \frac{s_{n_1}(0, a_1)}{s_{n_1}(1, a_1)} = \frac{a_1^{n_1} / n_1!}{\sum_{r=0}^{n_1} a_1^r / r!}. \quad (20)$$

We remark that the quantities L_1 , L_2 and I_{12} may be obtained directly, once the constants c_m are known, without calculating the probabilities p_{ij} .

III. SECONDARY QUEUE ONLY

We next consider the overflow problem in which there are $q_2 \geq 1$ waiting spaces in the secondary, but there is no primary queue. Now, if all n_2 servers in the secondary are busy when a demand from S_2 arrives, the demand is queued if one of the q_2 waiting spaces is available; otherwise, it is lost. Demands waiting in the secondary queue enter service (in some prescribed order) as servers in the secondary become free. If all n_1 servers in the primary are busy when a demand from S_1 arrives, and if there is a free server in the secondary and no demands are waiting in the secondary queue, then the demand is served in the secondary. If there are no free servers, the demand is lost.

Again, we let p_{ij} denote the steady-state probability that there are i demands in the primary and j demands in the secondary, either in service or waiting. It is convenient to introduce the function

$$\chi_\ell = \begin{cases} 1, & \ell \geq 0, \\ 0, & \ell < 0. \end{cases} \quad (21)$$

Then, with $k_2 = n_2 + q_2$, it is found that

$$\begin{aligned} & [a_1(1 - \delta_{in_1}\chi_{j-n_2}) + a_2(1 - \delta_{jk_2}) + i + \min(j, n_2)]p_{ij} \\ & = a_1(1 - \delta_{i0})p_{i-1,j} + (1 - \delta_{j0})(a_1\delta_{in_1}\chi_{n_2-j} + a_2)p_{i,j-1} \\ & + (1 - \delta_{in_1})(i+1)p_{i+1,j} + (1 - \delta_{jk_2})\min(j+1, n_2)p_{i,j+1} \end{aligned} \quad (22)$$

for $0 \leq i \leq n_1, 0 \leq j \leq k_2$. The normalization condition is

$$\sum_{i=0}^{n_1} \sum_{j=0}^{k_2} p_{ij} = 1. \quad (23)$$

If $i \neq n_1$, the variables in (22) may be separated, and there are solutions of the form $\alpha_i \beta_j$, where α_i satisfies (6) for $0 \leq i \leq n_1 - 1$ and

$$\begin{aligned} & [a_2(1 - \delta_{jk_2}) + \min(j, n_2) - \rho]\beta_j \\ & = a_2(1 - \delta_{j0})\beta_{j-1} + (1 - \delta_{jk_2})\min(j+1, n_2)\beta_{j+1} \end{aligned} \quad (24)$$

for $0 \leq j \leq k_2$. As before, $\alpha_i = s_i(\rho, a_1)$ for $0 \leq i \leq n_1$. For $0 \leq j \leq n_2 - 1$, (24) reduces to (7), and so there are solutions β_j proportional to $s_j(-\rho, a_2)$ for $0 \leq j \leq n_2$. Also, for $n_2 \leq j \leq k_2$,

$$[a_2(1 - \delta_{jk_2}) + n_2 - \rho]\beta_j = a_2\beta_{j-1} + n_2(1 - \delta_{jk_2})\beta_{j+1}. \quad (25)$$

The solution of (25) may be expressed in terms of Chebyshev polynomials of the second kind,⁹ $U_r(x)$. It is convenient to define

$$\Psi_r(\rho) = \left(\frac{n_2}{a_2}\right)^{r/2} U_r\left(\frac{a_2 + n_2 - \rho}{2\sqrt{a_2 n_2}}\right) \quad (26)$$

and

$$\phi_j(\rho) = \Psi_{k_2-j}(\rho) - \Psi_{k_2-j-1}(\rho). \quad (27)$$

The properties of these functions which we will need are given in Appendix A. We note here, however, that $U_0(x) \equiv 1$, $U_{-1}(x) \equiv 0$ and $\phi_{k_2}(\rho) \equiv 1$. It follows from (25) and (177) that β_j is proportional to $\phi_j(\rho)$ for $n_2 - 1 \leq j \leq k_2$.

Consequently, we take

$$\beta_j = \begin{cases} s_j(-\rho, a_2)\phi_{n_2}(\rho), & 0 \leq j \leq n_2, \\ s_{n_2}(-\rho, a_2)\phi_j(\rho), & n_2 - 1 \leq j \leq k_2, \end{cases} \quad (28)$$

where

$$s_{n_2-1}(-\rho, a_2)\phi_{n_2}(\rho) = s_{n_2}(-\rho, a_2)\phi_{n_2-1}(\rho). \quad (29)$$

This equation may be written in the form

$$\rho[s_{n_2}(1 - \rho, a_2)\Psi_{q_2}(\rho) - s_{n_2-1}(1 - \rho, a_2)\Psi_{q_2-1}(\rho)] = 0. \quad (30)$$

Note that this equation reduces to (9) if $q_2 = 0$. The expression in the square brackets in (30) is a polynomial in ρ of degree $k_2 = n_2 + q_2$. Appendix B shows that its zeros are positive and distinct, and we denote them by ρ_m , $m = 1, \dots, k_2$. As before, we let $\rho_0 = 0$.

It follows that we may represent the probabilities p_{ij} , for $0 \leq i \leq n_1$, in the form

$$p_{ij} = \begin{cases} \sum_{m=0}^{k_2} c_m s_i(\rho_m, \alpha_1) s_j(-\rho_m, \alpha_2) \phi_{n_2}(\rho_m), & 0 \leq j \leq n_2, \\ \sum_{m=0}^{k_2} c_m s_i(\rho_m, \alpha_1) s_{n_2}(-\rho_m, \alpha_2) \phi_j(\rho_m), & n_2 \leq j \leq k_2, \end{cases} \quad (31)$$

where the coefficients c_m are to be determined. Note that (31) reduces to (11) if $q_2 = 0$, since $\phi_{k_2}(\rho) \equiv 1$. It remains to satisfy the boundary conditions corresponding to $i = n_1$ in (22), as well as the normalization condition (23). Hence, in addition to (12) for $0 \leq j \leq n_2 - 1$, we must have

$$(\alpha_2 + n_1 + n_2)p_{n_1, n_2} = \alpha_1 p_{n_1-1, n_2} + (\alpha_1 + \alpha_2)p_{n_1, n_2-1} + n_2 p_{n_1, n_2+1} \quad (32)$$

and

$$\begin{aligned} [\alpha_2(1 - \delta_{jk_2}) + n_1 + n_2]p_{n_1, j} \\ = \alpha_1 p_{n_1-1, j} + \alpha_2 p_{n_1, j-1} + n_2(1 - \delta_{jk_2})p_{n_1, j+1} \end{aligned} \quad (33)$$

for $n_2 + 1 \leq j \leq k_2$.

If we substitute (31) into (12), we find, after reduction, that

$$\begin{aligned} \sum_{m=0}^{k_2} c_m [\rho_m s_{n_1}(1 + \rho_m, \alpha_1) s_j(-\rho_m, \alpha_2) \\ + \alpha_1 s_{n_1}(\rho_m, \alpha_1) s_j(-1 - \rho_m, \alpha_2)] \phi_{n_2}(\rho_m) = 0, \end{aligned} \quad (34)$$

for $0 \leq j \leq n_2 - 1$. Also, from (33) we obtain

$$\sum_{m=0}^{k_2} c_m \rho_m s_{n_1}(1 + \rho_m, \alpha_1) s_{n_2}(-\rho_m, \alpha_2) \phi_j(\rho_m) = 0 \quad (35)$$

for $n_2 + 1 \leq j \leq k_2$, and from (32) we find that

$$\begin{aligned} \sum_{m=0}^{k_2} c_m [\rho_m s_{n_1}(1 + \rho_m, \alpha_1) s_{n_2}(-\rho_m, \alpha_2) \\ - \alpha_1 s_{n_1}(\rho_m, \alpha_1) s_{n_2-1}(-\rho_m, \alpha_2)] \phi_{n_2}(\rho_m) = 0. \end{aligned} \quad (36)$$

This last condition is redundant, as may be verified by summing (34)

from $j = 0$ to $n_2 - 1$ and (35) from $j = n_2 + 1$ to k_2 , and using the fact that ρ_m satisfies (30). Hence, the coefficients c_m are determined by (34) and (35) only to within a multiplicative constant, which is determined from the normalization condition (23). From (23) and (31), with the help of (27), (30), and (170) and (171) of Appendix A, it is found that

$$c_0 s_{n_1}(1, a_1) [s_{n_2}(1, a_2) \Psi_{q_2}(0) - s_{n_2-1}(1, a_2) \Psi_{q_2-1}(0)] = 1. \quad (37)$$

We note, from (167), (168), and (176), that

$$s_n(1, a) = \sum_{r=0}^n \frac{a^r}{r!}, \quad \Psi_\ell(0) = \sum_{r=0}^{\ell} \left(\frac{n_2}{a_2} \right)^r. \quad (38)$$

Once the constants c_m have been determined, the probabilities p_{ij} may be calculated from (31). In particular, with the help of (27), (30), (37), (167), and (170), it is found that

$$\sum_{j=0}^{k_2} p_{ij} = \frac{s_i(0, a_1)}{s_{n_1}(1, a_1)} = \frac{a_1^i / i!}{\sum_{r=0}^{n_1} a_1^r / r!}, \quad (39)$$

as is to be expected. Moreover, the steady-state quantities of interest may be calculated. If L_k is the probability that a demand from S_k is lost, then

$$L_1 = \sum_{j=n_2}^{k_2} p_{n_1, j}, \quad L_2 = \sum_{i=0}^{n_1} p_{i, k_2}. \quad (40)$$

Also, the probability of overflow I_{12} is as given by (5), and so, from (39),

$$L_1 + I_{12} = \sum_{j=0}^{k_2} p_{n_1, j} = \frac{s_{n_1}(0, a_1)}{s_{n_1}(1, a_1)} = \frac{a_1^{n_1} / n_1!}{\sum_{r=0}^{n_1} a_1^r / r!}. \quad (41)$$

From (31), it is found that

$$L_2 = \sum_{m=0}^{k_2} c_m s_{n_1}(1 + \rho_m, a_1) s_{n_2}(-\rho_m, a_2), \quad (42)$$

since $\phi_{k_2}(\rho) \equiv 1$, from (27), and

$$I_{12} = \sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, a_1) s_{n_2-1}(1 - \rho_m, a_2) \phi_{n_2}(\rho_m). \quad (43)$$

Other quantities of interest are the probability that a demand from the secondary source is queued on arrival, namely,

$$Q_2 = \sum_{i=0}^{n_1} \sum_{j=n_2}^{k_2-1} p_{ij}, \quad (44)$$

and the average number of demands in the secondary queue,

$$V_2 = \sum_{i=0}^{n_1} \sum_{j=n_2+1}^{k_2} (j - n_2) p_{ij}. \quad (45)$$

But, if we let

$$r_j = \sum_{i=0}^{n_1} p_{ij}, \quad (46)$$

and sum on i in (22), we obtain

$$[a_2(1 - \delta_{jk_2}) + n_2]r_j = a_2 r_{j-1} + n_2(1 - \delta_{jk_2})r_{j+1}, \quad (47)$$

for $n_2 + 1 \leq j \leq k_2$. It follows that

$$n_2 r_j = a_2 r_{j-1}, \quad n_2 + 1 \leq j \leq k_2. \quad (48)$$

Hence, since $L_2 = r_{k_2}$, from (40) and (46),

$$r_j = \left(\frac{n_2}{a_2}\right)^{k_2-j} L_2, \quad n_2 \leq j \leq k_2. \quad (49)$$

From (45), (46), and (49), we obtain

$$V_2 = \Delta_{q_2} \left(\frac{n_2}{a_2}\right) L_2, \quad (50)$$

where

$$\Delta_q(\xi) = \sum_{\ell=1}^q \ell \xi^{q-\ell} = \begin{cases} [q - (q+1)\xi + \xi^{q+1}]/(1-\xi)^2, & \xi \neq 1, \\ \frac{1}{2} q(q+1), & \xi = 1. \end{cases} \quad (51)$$

Similarly, from (44),

$$Q_2 = \Gamma_{q_2} \left(\frac{n_2}{a_2}\right) L_2, \quad (52)$$

where

$$\Gamma_q(\xi) = \sum_{\ell=0}^{q-1} \xi^{q-\ell} = q + (\xi - 1)\Delta_q(\xi). \quad (53)$$

The departure rate from the secondary queue is

$$R_{22} = n_2 \mu \sum_{i=0}^{n_1} \sum_{j=n_2+1}^{k_2} p_{ij} = n_2 \mu \sum_{j=n_2+1}^{k_2} r_j, \quad (54)$$

from (46). It may be verified from (49) that $R_{22} = \lambda_2 Q_2$. This result holds since, in the steady state, the departure rate from the secondary queue is equal to the arrival rate to it.

Now, according to Little's theorem,¹⁰ the average number of demands in a queuing system is equal to the average rate of arrival of demands to that system times the average time spent in that system. If we apply this result to the secondary queue, we find that the average

waiting time of the demands which are queued in the secondary is

$$W_2 = \frac{V_2}{\lambda_2 Q_2}, \quad (55)$$

independently of the order of service within the queue. Now the average number of demands being served by the secondary group is

$$X_2 = \sum_{i=0}^{n_1} \sum_{j=0}^{k_2} \min(j, n_2) p_{ij}. \quad (56)$$

Hence, if we apply Little's theorem to the secondary group of servers, we obtain

$$a_2(1 - L_2) + a_1 I_{12} = X_2, \quad (57)$$

from (1), since the mean service rate is μ . From (31) and (56), with the help of (23) and (171) and (172) of Appendix A, it is found that

$$X_2 = n_2 - \sum_{m=0}^{k_2} c_m s_{n_1} (1 + \rho_m, a_1) s_{n_2-1} (2 - \rho_m, a_2) \phi_{n_2}(\rho_m). \quad (58)$$

The relationship (57) may be verified from (42), (43), and (58), with the help of the boundary conditions (34) and (35), although the verification is difficult. Consequently, the relationship (57) provides a useful check on the numerical calculation of the constants c_m .

IV. PRIMARY AND SECONDARY QUEUES: CASE I

We now begin the investigation of the overflow problem in which there is a primary queue, with $q_1 \geq 1$ waiting spaces, as well as a secondary queue, with q_2 waiting spaces. As before, if all n_2 servers in the secondary are busy when a demand from S_2 arrives, the demand is queued if one of the q_2 waiting spaces is available; otherwise, it is lost. Demands waiting in the secondary queue enter service as servers in the secondary become free. If all n_1 servers in the primary are busy when a demand from S_1 arrives, and if there is a free server in the secondary and no demands are waiting in the secondary queue, then the demand is served in the secondary. If there are no free servers, then the demand is queued in the primary queue, if one of the q_1 waiting spaces is available; otherwise, it is lost.

We will consider two different cases for the treatment of demands that are waiting in the primary queue. In the first case, to be considered now, a demand waiting in the primary queue may enter service either in the primary, when a server becomes free, or in the secondary, if a server becomes free and there are no demands waiting in the secondary queue. Thus overflow from the primary queue is permitted in the first case. In the second case, to be considered later, no overflow is permitted from the primary queue, so that a demand in the primary queue must wait for a server in the primary to become free.

We let

$$k_1 = n_1 + q_1, \quad k_2 = n_2 + q_2. \quad (59)$$

Then, in the first case, it is found that the steady-state probabilities p_{ij} satisfy the equations

$$\begin{aligned} & [a_1(1 - \delta_{ik_1}\chi_{j-n_2}) + a_2(1 - \delta_{jk_2}) + \min(i, n_1) + \min(j, n_2)]p_{ij} \\ &= (1 - \chi_{i-1-n_1}\chi_{n_2-1-j})[a_1(1 - \delta_{i0})p_{i-1,j} \\ & \quad + (1 - \delta_{jk_2})\min(j+1, n_2)p_{i,j+1}] \\ & \quad + (1 - \delta_{j0})[a_1\delta_{in_1}\chi_{n_2-j} + a_2(1 - \chi_{i-1-n_1}\chi_{n_2-j})]p_{i,j-1} \\ & \quad + (1 - \delta_{ik_1})[(1 - \chi_{i-n_1}\chi_{n_2-1-j})\min(i+1, n_1) + n_2\chi_{i-n_1}\delta_{jn_2}]p_{i+1,j}, \end{aligned} \quad (60)$$

for $0 \leq i \leq k_1$, $0 \leq j \leq k_2$. These equations were constructed so as to imply that

$$p_{ij} = 0, \quad n_1 + 1 \leq i \leq k_1, \quad 0 \leq j \leq n_2 - 1, \quad (61)$$

since it is impossible for demands to be waiting in the primary queue when there is a server free in the secondary. The normalization condition is

$$\sum_{i=0}^{n_1} \sum_{j=0}^{k_2} p_{ij} + \sum_{i=n_1+1}^{k_1} \sum_{j=n_2}^{k_2} p_{ij} = 1. \quad (62)$$

We assume for the time being that $q_2 \geq 1$. For $0 \leq i \leq n_1 - 1$ and $0 \leq j \leq k_2$, (60) is equivalent to (22), and hence there are solutions of the form $\alpha_i \beta_j$, where $\alpha_i = s_i(\rho, a_1)$ for $0 \leq i \leq n_1$, β_j is given by (28), and ρ is a root of equation (30). For $n_1 + 1 \leq i \leq k_1$ and $n_2 + 1 \leq j \leq k_2$, the variables in (60) may be separated, and there are solutions of the form $\gamma_i \delta_j$, where

$$\begin{aligned} & [a_1(1 - \delta_{ik_1}) + n_1 + \nu]\gamma_i \\ &= a_1 \gamma_{i-1} + n_1(1 - \delta_{ik_1})\gamma_{i+1}, \quad n_1 + 1 \leq i \leq k_1 \end{aligned} \quad (63)$$

and

$$\begin{aligned} & [a_2(1 - \delta_{jk_2}) + n_2 - \nu]\delta_j \\ &= a_2\delta_{j-1} + n_2(1 - \delta_{jk_2})\delta_{j+1}, \quad n_2 + 1 \leq j \leq k_2, \end{aligned} \quad (64)$$

and ν is a separation constant. Corresponding to (26) and (27), we define

$$\Omega_\ell(\nu) = \left(\frac{n_1}{a_1}\right)^{\ell/2} U_\ell\left(\frac{a_1 + n_1 + \nu}{2\sqrt{a_1 n_1}}\right) \quad (65)$$

and

$$\theta_i(\nu) = \Omega_{k_1-i}(\nu) - \Omega_{k_1-i-1}(\nu). \quad (66)$$

Then γ_i is proportional to $\theta_i(\nu)$ for $n_1 \leq i \leq k_1$ and δ_j is proportional to $\phi_j(\nu)$ for $n_2 \leq j \leq k_2$. We note that $\theta_{k_1}(\nu) \equiv 1$.

Now $\theta_{n_1}(\nu)$ is a polynomial of degree q_1 in ν . It is shown in Appendix B that its zeros are negative and distinct, and we denote them by ν_ℓ , $\ell = 1, \dots, q_1$. Then

$$\theta_{n_1}(\nu_\ell) = 0, \quad \ell = 1, \dots, q_1. \quad (67)$$

Since the roots ρ_m , $m = 0, \dots, k_2$ of (30) are nonnegative and distinct, we have a joint set of distinct roots. In view of the above results, we may represent the probabilities p_{ij} in the form

$$p_{ij} = \begin{cases} \sum_{m=0}^{k_2} c_m s_i(\rho_m, a_1) s_j(-\rho_m, a_2) \theta_{n_1}(\rho_m) \phi_{n_2}(\rho_m), & 0 \leq j \leq n_2, \\ \sum_{m=0}^{k_2} c_m s_i(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \theta_{n_1}(\rho_m) \phi_j(\rho_m), & n_2 \leq j \leq k_2, \end{cases} \quad (68)$$

for $0 \leq i \leq n_1$, and

$$p_{ij} = \sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \theta_i(\rho_m) \phi_j(\rho_m) + \sum_{\ell=1}^{q_1} b_\ell \theta_i(\nu_\ell) \phi_j(\nu_\ell), \quad n_1 \leq i \leq k_1, n_2 \leq j \leq k_2, \quad (69)$$

where the constants c_m and b_ℓ are to be determined. Note that the representations in (68) and (69) agree for $i = n_1$, in view of (67). We also note that (68) reduces to (31) if $q_1 = 0$, since $\theta_{k_1}(\nu) \equiv 1$.

It remains to satisfy the boundary conditions at $i = n_1$, $0 \leq j \leq k_2$ and at $j = n_2$, $n_1 + 1 \leq i \leq k_1$, as well as the normalization condition (62). Hence, in addition to (12) for $0 \leq j \leq n_2 - 1$, we must have

$$(a_1 + a_2 + n_1 + n_2)p_{n_1, n_2} = a_1 p_{n_1-1, n_2} + (a_1 + a_2)p_{n_1, n_2-1} + (n_1 + n_2)p_{n_1+1, n_2} + n_2 p_{n_1, n_2+1} \quad (70)$$

and

$$[a_1 + a_2(1 - \delta_{jk_2}) + n_1 + n_2]p_{n_1, j} = a_1 p_{n_1-1, j} + a_2 p_{n_1, j-1} + n_1 p_{n_1+1, j} + n_2(1 - \delta_{jk_2})p_{n_1, j+1} \quad (71)$$

for $n_2 + 1 \leq j \leq k_2$. Also,

$$[a_1(1 - \delta_{ik_1}) + a_2 + n_1 + n_2]p_{i, n_2} = a_1 p_{i-1, n_2} + (n_1 + n_2)(1 - \delta_{ik_1})p_{i+1, n_2} + n_2 p_{i, n_2+1} \quad (72)$$

for $n_1 + 1 \leq i \leq k_1$.

If we substitute (68) into (12), we find, after reduction, that

$$\sum_{m=0}^{k_2} c_m [\rho_m s_{n_1}(1 + \rho_m, a_1) s_j(-\rho_m, a_2) + a_1 s_{n_1}(\rho_m, a_1) s_j(-1 - \rho_m, a_2)] \theta_{n_1}(\rho_m) \phi_{n_2}(\rho_m) = 0, \quad (73)$$

for $0 \leq j \leq n_2 - 1$. Next, from (68), (69), and (71), with the help of (184), we obtain

$$\sum_{m=0}^{k_2} c_m \rho_m s_{n_2}(-\rho_m, a_2) [s_{n_1}(1 + \rho_m, a_1) \theta_{n_1}(\rho_m) + s_{n_1}(\rho_m, a_1) \Omega_{q_1-1}(\rho_m)] \phi_j(\rho_m) = n_1 \sum_{\ell=1}^{q_1} b_\ell \theta_{n_1+1}(\nu_\ell) \phi_j(\nu_\ell), \quad (74)$$

for $n_2 + 1 \leq j \leq k_2$. Also, from (72), with the help of the recurrence relation (63) satisfied by $\theta_i(\nu)$, it is found that

$$\begin{aligned} \sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) [a_2 \theta_i(\rho_m) \phi_{n_2-1}(\rho_m) \\ - n_2(1 - \delta_{ik_1}) \theta_{i+1}(\rho_m) \phi_{n_2}(\rho_m)] \\ + \sum_{\ell=1}^{q_1} b_\ell [a_2 \theta_i(\nu_\ell) \phi_{n_2-1}(\nu_\ell) - n_2(1 - \delta_{ik_1}) \theta_{i+1}(\nu_\ell) \phi_{n_2}(\nu_\ell)] = 0, \quad (75) \end{aligned}$$

for $n_1 + 1 \leq i \leq k_1$. It may be verified that the boundary condition (70) is redundant.

The coefficients c_m and b_ℓ are determined by (73) to (75) only to within a multiplicative constant, which is determined from the normalization condition (62). But, from (68), with the help of (27), (30), and (170) and (171) of Appendix A, it is found that

$$\sum_{j=0}^{k_2} p_{ij} = c_0 s_i(0, a_1) \theta_{n_1}(0) [s_{n_2}(1, a_2) \Psi_{q_2}(0) - s_{n_2-1}(1, a_2) \Psi_{q_2-1}(0)] \quad (76)$$

for $0 \leq i \leq n_1$. Then, from (62) and (69), with the help of (66) and (171), it follows that

$$\begin{aligned} c_0 s_{n_1}(1, a_1) \theta_{n_1}(0) [s_{n_2}(1, a_2) \Psi_{q_2}(0) - s_{n_2-1}(1, a_2) \Psi_{q_2-1}(0)] \\ + \sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \Omega_{q_1-1}(\rho_m) \Psi_{q_2}(\rho_m) \\ + \sum_{\ell=1}^{q_1} b_\ell \Omega_{q_1-1}(\nu_\ell) \Psi_{q_2}(\nu_\ell) = 1. \quad (77) \end{aligned}$$

Note that (77) reduces to (37) if $q_1 = 0$, since $\Omega_{-1}(\nu) \equiv 0$ and $\theta_{k_1}(\nu) \equiv 1$. Once the constants c_m and b_ℓ have been determined, the steady-state probabilities p_{ij} may be calculated from (68) and (69). We remark that the number of constants to be determined is only $k_2 + 1 + q_1$, whereas

the number of probabilities p_{ij} is $(n_1 + 1)(k_2 + 1) + q_1(q_2 + 1)$, which in general is considerably larger.

V. SOME STEADY-STATE QUANTITIES: CASE I

Other steady-state quantities of interest are depicted in Fig. 1, which indicates the mean flow rates. The loss probabilities L_1 and L_2 are given by

$$L_1 = \sum_{j=n_2}^{k_2} p_{k_1, j}, \quad L_2 = \sum_{i=0}^{k_1} p_{i, k_2}, \quad (78)$$

and the probabilities that a demand from the primary, or secondary, source is queued on arrival are

$$Q_1 = \sum_{i=n_1}^{k_1-1} \sum_{j=n_2}^{k_2} p_{ij}, \quad Q_2 = \sum_{i=0}^{k_1} \sum_{j=n_2}^{k_2-1} p_{ij}. \quad (79)$$

The probability that a demand arriving from the primary source overflows immediately is I_{12} , as given by (5).

Since the mean service rate is μ , the mean departure rate from the primary queue to the primary servers is

$$R_{11} = n_1 \mu \sum_{i=n_1+1}^{k_1} \sum_{j=n_2}^{k_2} p_{ij}, \quad (80)$$

while the mean rate of overflow from the primary queue to the secondary servers is

$$R_{12} = n_2 \mu \sum_{i=n_1+1}^{k_1} p_{i, n_2}. \quad (81)$$

The mean departure rate from the secondary queue is

$$R_{22} = n_2 \mu \sum_{i=0}^{k_1} \sum_{j=n_2+1}^{k_2} p_{ij}. \quad (82)$$

It may be verified from (60) that

$$R_{11} + R_{12} = \lambda_1 Q_1, \quad R_{22} = \lambda_2 Q_2. \quad (83)$$

The average number of demands in the primary and secondary queues are

$$V_1 = \sum_{i=n_1+1}^{k_1} \sum_{j=n_2}^{k_2} (i - n_1) p_{ij}, \quad V_2 = \sum_{i=0}^{k_1} \sum_{j=n_2+1}^{k_2} (j - n_2) p_{ij}. \quad (84)$$

Also, the average number of demands in service in the two groups are

$$X_1 = \sum_{i=0}^{n_1} \sum_{j=0}^{k_2} i p_{ij} + n_1 \sum_{i=n_1+1}^{k_1} \sum_{j=n_2}^{k_2} p_{ij} \quad (85)$$

and

$$X_2 = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2-1} j p_{ij} + n_2 \sum_{i=0}^{k_1} \sum_{j=n_2}^{k_2} p_{ij}. \quad (86)$$

If we apply Little's theorem to the primary and secondary queues, we find that the average waiting times of the demands which are queued in the primary or in the secondary are given by

$$W_1 = \frac{V_1}{\lambda_1 Q_1}, \quad W_2 = \frac{V_2}{\lambda_2 Q_2}, \quad (87)$$

respectively, independently of the order of service within each queue. Also, if we apply Little's theorem to the primary and secondary groups of servers, we obtain

$$\lambda_1(1 - L_1 - I_{12}) - R_{12} = \mu X_1, \quad \lambda_2(1 - L_2) + \lambda_1 I_{12} + R_{12} = \mu X_2, \quad (88)$$

since the mean service rate is μ .

The steady-state quantities of interest may be expressed in terms of the constants c_m and b_ℓ by use of the representations in (68) and (69). From (78), with the help of (27), since $\theta_{k_1}(\nu) \equiv 1$, it is found that

$$L_1 = \sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, \alpha_1) s_{n_2}(-\rho_m, \alpha_2) \Psi_{q_2}(\rho_m) + \sum_{\ell=1}^{q_1} b_\ell \Psi_{q_2}(\nu_\ell), \quad (89)$$

and, with the help of (66) and (171), since $\phi_{k_2}(\rho) \equiv 1$, it is found that

$$L_2 = \sum_{m=0}^{k_2} c_m s_{n_2}(-\rho_m, \alpha_2) [s_{n_1}(1 + \rho_m, \alpha_1) \theta_{n_1}(\rho_m) + s_{n_1}(\rho_m, \alpha_1) \Omega_{q_1-1}(\rho_m)] + \sum_{\ell=1}^{q_1} b_\ell \Omega_{q_1-1}(\nu_\ell). \quad (90)$$

Similarly, it follows from (79) that

$$Q_1 = \sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, \alpha_1) s_{n_2}(-\rho_m, \alpha_2) [\Omega_{q_1}(\rho_m) - 1] \Psi_{q_2}(\rho_m) + \sum_{\ell=1}^{q_1} b_\ell [\Omega_{q_1}(\nu_\ell) - 1] \Psi_{q_2}(\nu_\ell). \quad (91)$$

Next, from (5), with the help of (171), it is found that

$$I_{12} = \sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, \alpha_1) s_{n_2-1}(1 - \rho_m, \alpha_2) \theta_{n_1}(\rho_m) \phi_{n_2}(\rho_m). \quad (92)$$

Also, from (80) and (81), it follows that

$$R_{11} = n_1 \mu \left[\sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, \alpha_1) s_{n_2}(-\rho_m, \alpha_2) \Omega_{q_1-1}(\rho_m) \Psi_{q_2}(\rho_m) + \sum_{\ell=1}^{q_1} b_\ell \Omega_{q_1-1}(\nu_\ell) \Psi_{q_2}(\nu_\ell) \right], \quad (93)$$

and

$$R_{12} = n_2 \mu \left[\sum_{m=0}^{k_2} c_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \Omega_{q_1-1}(\rho_m) \phi_{n_2}(\rho_m) + \sum_{\ell=1}^{q_1} b_\ell \Omega_{q_1-1}(\nu_\ell) \phi_{n_2}(\nu_\ell) \right]. \quad (94)$$

The first relationship in (83) provides a useful numerical check, in view of the expressions in (91), (93), and (94).

From (84), with the help of (27), (182), and (183), it is found that

$$\begin{aligned} V_1 &= c_0 s_{n_1}(0, a_1) s_{n_2}(0, a_2) \Psi_{q_2}(0) \Delta_{q_1} \left(\frac{n_1}{a_1} \right) \\ &\quad - a_1 \sum_{m=1}^{k_2} \frac{c_m}{\rho_m} s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \Psi_{q_2}(\rho_m) \\ &\quad \cdot \left[1 - \Omega_{q_1}(\rho_m) + \frac{n_1}{a_1} \Omega_{q_1-1}(\rho_m) \right] \\ &\quad - a_1 \sum_{\ell=1}^{q_1} \frac{b_\ell}{\nu_\ell} \Psi_{q_2}(\nu_\ell) \left[1 - \Omega_{q_1}(\nu_\ell) + \frac{n_1}{a_1} \Omega_{q_1-1}(\nu_\ell) \right], \end{aligned} \quad (95)$$

where $\Delta_q(\xi)$ is as defined in (51). From (85), with the help of (62), (76), and (172), it follows that

$$\begin{aligned} X_1 &= n_1 - c_0 s_{n_1-1}(2, a_1) \theta_{n_1}(0) \\ &\quad \cdot [s_{n_2}(1, a_2) \Psi_{q_2}(0) - s_{n_2-1}(1, a_2) \Psi_{q_2-1}(0)]. \end{aligned} \quad (96)$$

Similarly, from (86), with the help of (62), (68), (171), and (172),

$$X_2 = n_2 - \sum_{m=0}^{k_2} c_m s_{n_1}(1 + \rho_m, a_1) s_{n_2-1}(2 - \rho_m, a_2) \theta_{n_1}(\rho_m) \phi_{n_2}(\rho_m). \quad (97)$$

In view of (89), (90), (92), (94), (96), and (97), the relationships in (88) provide a useful numerical check.

If we now define

$$r_j = \sum_{i=0}^{k_1} p_{ij}, \quad (98)$$

and sum on i in (60), for $n_2 + 1 \leq j \leq k_2$, we obtain (47), and hence (48). Also, $L_2 = r_{k_2}$, from (78) and (98), and it follows that (49) holds and, from (79) and (84), that (50) and (52) hold. Moreover, from (82) and (98),

$$R_{22} = n_2 \mu \sum_{j=n_2+1}^{k_2} r_j, \quad (99)$$

and it may be verified, as before, that $R_{22} = \lambda_2 Q_2$. This completes the calculation of expressions for the steady-state quantities of interest.

We have assumed that $q_2 \geq 1$. We remark, however, that the representations in (68) and (69) remain valid in the case $q_2 = 0$, so that $k_2 = n_2$ and $\phi_{n_2}(\rho) \equiv 1$. We have verified that (73) and (75) to (77) hold for $q_2 = 0$, and note that $\Psi_0(\rho) \equiv 1$ and $\Psi_{-1}(\rho) \equiv 0$. We have also verified that (89) to (97) hold for $q_2 = 0$. Moreover, $V_2 = 0$ and $Q_2 = 0$, which is consistent with (50) and (52), as is seen from (51) and (53). In the next section, we investigate the case $q_2 = 0$ in a different manner.

VI. ANOTHER APPROACH: CASE I

We now give special consideration to the case $q_1 \geq 1$, $q_2 = 0$, so that $k_2 = n_2$. Then (60) for $0 \leq i \leq n_1 - 1$ and $0 \leq j \leq n_2$ leads to the representation

$$p_{ij} = \sum_{m=0}^{n_2} d_m s_i(\rho_m, a_1) s_j(-\rho_m, a_2), \quad 0 \leq i \leq n_1, \quad 0 \leq j \leq n_2, \quad (100)$$

corresponding to (11), where ρ_m , $m = 0, \dots, n_2$ are the roots of (10), with $\rho_0 = 0$. The coefficients d_m are to be determined. The only other nonzero steady-state probabilities are p_{i,n_2} for $n_1 + 1 \leq i \leq k_1$. But, from (60), with $k_2 = n_2$,

$$[a_1(1 - \delta_{ik_1}) + n_1 + n_2]p_{i,n_2} = a_1 p_{i-1,n_2} + (n_1 + n_2)(1 - \delta_{ik_1})p_{i+1,n_2} \quad (101)$$

for $n_1 + 1 \leq i \leq k_1$. It follows that

$$(n_1 + n_2)p_{i,n_2} = a_1 p_{i-1,n_2}, \quad n_1 + 1 \leq i \leq k_1. \quad (102)$$

Hence,

$$p_{i,n_2} = \left(\frac{a_1}{n_1 + n_2} \right)^{i-n_1} p_{n_1,n_2}, \quad n_1 \leq i \leq k_1, \quad (103)$$

and p_{n_1,n_2} is given by (100). We remark that (103) holds if $q_1 = \infty$, provided that the stability condition $a_1 < n_1 + n_2$ is satisfied.

It remains to satisfy the boundary conditions at $i = n_1$, $0 \leq j \leq n_2$, as well as the normalization condition (62) with $k_2 = n_2$. But (60) implies that (12) must hold for $0 \leq j \leq n_2 - 1$, and from (100) it follows that, corresponding to (13),

$$\sum_{m=0}^{n_2} d_m [\rho_m s_{n_1}(1 + \rho_m, a_1) s_j(-\rho_m, a_2) + a_1 s_{n_1}(\rho_m, a_1) s_j(-1 - \rho_m, a_2)] = 0, \quad (104)$$

for $0 \leq j \leq n_2 - 1$. It may be verified that the boundary condition corresponding to $i = n_1$ and $j = n_2$, namely,

$$(a_1 + n_1 + n_2)p_{n_1,n_2} = a_1 p_{n_1-1,n_2} + (a_1 + a_2)p_{n_1,n_2-1} + (n_1 + n_2)p_{n_1+1,n_2}, \quad (105)$$

is redundant. In fact, (105) reduces to (14), which was shown to be redundant, since

$$(n_1 + n_2)p_{n_1+1, n_2} = a_1 p_{n_1, n_2} \quad (106)$$

from (103).

The coefficients d_m are determined by (104) only to within a multiplicative constant, which is determined by the normalization condition. From (62), with $k_2 = n_2$, it is found, with the help of (10), (53), (100), (103), and (171), that

$$d_0 s_{n_1}(1, a_1) s_{n_2}(1, a_2) + \left(\frac{a_1}{n_1 + n_2} \right)^{q_1+1} \Gamma_{q_1} \left(\frac{n_1 + n_2}{a_1} \right) \cdot \sum_{m=0}^{n_2} d_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) = 1. \quad (107)$$

We note that, for $a_1 < n_1 + n_2$,

$$\lim_{q_1 \rightarrow \infty} \left(\frac{a_1}{n_1 + n_2} \right)^{q_1+1} \Gamma_{q_1} \left(\frac{n_1 + n_2}{a_1} \right) = \frac{a_1}{(n_1 + n_2 - a_1)}. \quad (108)$$

Once the coefficients d_m have been determined from (104) and (107), the steady-state quantities of interest may be calculated with the help of (100) and (103). Since the calculation of these quantities is a straightforward extension of previous calculations, in view of the simplicity of the result in (103), we do not present the results here. We remark that the approach adopted here in the case $q_2 = 0$ requires the determination of only $n_2 + 1$ constants d_m , $m = 0, \dots, n_2$, whereas the approach adopted in the two previous sections requires the determination of the $n_2 + 1$ constants c_m , $m = 0, \dots, n_2$ and the q_1 constants b_ℓ , $\ell = 1, \dots, q_1$. This raises the question of an analogous approach for $q_2 \geq 1$. We have carried out the corresponding analysis and will present the results in a separate paper. The results are useful, in particular, when q_1 is large, or even infinite.

VII. PRIMARY QUEUE ONLY: CASE II

We now begin the investigation of the second case, in which no overflow is permitted from the primary queue, so that a demand in the primary queue must wait for a server in the primary to become free. Hence, $R_{12} \equiv 0$ in the diagram in Fig. 1, and this is the only aspect in which the second case differs from the first. In this section, we consider the case in which there is a primary queue only, with $q_1 \geq 1$ waiting spaces. The steady-state probabilities p_{ij} now satisfy the equations

$$\begin{aligned} & [a_1(1 - \delta_{ik_1}\delta_{jn_2}) + a_2(1 - \delta_{jn_2}) + \min(i, n_1) + j]p_{ij} \\ &= a_1(1 - \delta_{i0})[1 - \chi_{i-n_1-1}(1 - \delta_{jn_2})]p_{i-1,j} \\ &+ (1 - \delta_{j0})(a_1\chi_{i-n_1} + a_2)p_{i,j-1} \\ &+ (1 - \delta_{ik_1})\min(i+1, n_1)p_{i+1,j} + (1 - \delta_{jn_2})(j+1)p_{i,j+1}, \end{aligned} \quad (109)$$

for $0 \leq i \leq k_1$, $0 \leq j \leq n_2$, where $k_1 = n_1 + q_1$. The normalization condition is

$$\sum_{i=0}^{k_1} \sum_{j=0}^{n_2} p_{ij} = 1. \quad (110)$$

But (109) for $0 \leq i \leq n_1 - 1$ and $0 \leq j \leq n_2$ leads, analogously to case I, to the representation

$$p_{ij} = \sum_{m=0}^{n_2} D_m s_i(\rho_m, a_1) s_j(-\rho_m, a_2), \quad 0 \leq i \leq n_1, \quad 0 \leq j \leq n_2, \quad (111)$$

where ρ_m , $m = 0, \dots, n_2$ are the roots of (10), with $\rho_0 = 0$. The coefficients D_m are to be determined. Also, for $n_1 + 1 \leq i \leq k_1$ and $0 \leq j \leq n_2 - 1$,

$$\begin{aligned} & (a_1 + a_2 + n_1 + j)p_{ij} \\ &= (a_1 + a_2)(1 - \delta_{j0})p_{i,j-1} + n_1(1 - \delta_{ik_1})p_{i+1,j} + (j+1)p_{i,j+1}. \end{aligned} \quad (112)$$

We define the quantities Π_{rj} , for $r, j = 0, 1, \dots$, as the solutions of the equations

$$\begin{aligned} & (a_1 + a_2 + n_1 + j)\Pi_{rj} \\ &= (a_1 + a_2)(1 - \delta_{j0})\Pi_{r,j-1} + n_1(1 - \delta_{r0})\Pi_{r-1,j} + (j+1)\Pi_{r,j+1}, \end{aligned} \quad (113)$$

which satisfy the initial conditions

$$\Pi_{r0} = \delta_{r0}, \quad r = 0, 1, \dots \quad (114)$$

It follows from (163) that

$$\sum_{r=0}^{\infty} \Pi_{rj} v^r = s_j(n_1(1-v), a_1 + a_2), \quad (115)$$

and, in particular,

$$\Pi_{0j} = s_j(n_1, a_1 + a_2). \quad (116)$$

Also, (115) implies that

$$\Pi_{rj} = 0, \quad r > j. \quad (117)$$

It may be verified, from (112), (113), and (117), that

$$p_{ij} = \sum_{r=i}^{k_1} p_{r0} \Pi_{r-i,j}, \quad n_1 + 1 \leq i \leq k_1, \quad 0 \leq j \leq n_2. \quad (118)$$

The quantities Π_{rj} may be calculated sequentially from the recurrence relations (113), with the help of (117) and the initial condition $\Pi_{00} = 1$.

It remains to satisfy the boundary conditions at $i = n_1$, $0 \leq j \leq n_2$ and at $j = n_2$, $n_1 + 1 \leq i \leq k_1$, as well as the normalization condition

(110). Hence, from (109) we must have

$$[a_1 + a_2(1 - \delta_{jn_2}) + n_1 + j]p_{n_1,j} = a_1p_{n_1-1,j} + (a_1 + a_2)(1 - \delta_{j0})p_{n_1,j-1} \\ + n_1p_{n_1+1,j} + (1 - \delta_{jn_2})(j+1)p_{n_1,j+1} \quad (119)$$

for $0 \leq j \leq n_2$ and

$$[a_1(1 - \delta_{ik_1}) + n_1 + n_2]p_{i,n_2} \\ = a_1p_{i-1,n_2} + (a_1 + a_2)p_{i,n_2-1} + n_1(1 - \delta_{ik_1})p_{i+1,n_2} \quad (120)$$

for $n_1 + 1 \leq i \leq k_1$.

From (111), (118), and (119), it is found, after reduction, that

$$\sum_{m=0}^{n_2} D_m[\rho_m s_{n_1}(1 + \rho_m, a_1)s_j(-\rho_m, a_2) + a_1 s_{n_1}(\rho_m, a_1)s_j(-1 - \rho_m, a_2)] \\ - n_1 \sum_{r=n_1+1}^{k_1} p_{r0} \Pi_{r-n_1-1,j} = 0 \quad (121)$$

for $0 \leq j \leq n_2 - 1$. Also, with the help of (113), the boundary conditions (120) may be written in the form

$$\sum_{r=n_1+1}^{k_1} p_{r0}[(n_2 + 1)\Pi_{r-n_1-1,n_2+1} - (a_1\delta_{n_1+1,k_1} + a_2)\Pi_{r-n_1-1,n_2}] \\ - a_1 \sum_{m=0}^{n_2} D_m s_{n_1}(\rho_m, a_1)s_{n_2}(-\rho_m, a_2) = 0, \quad (122)$$

and, for $q_1 \geq 2$,

$$\sum_{r=i}^{k_1} p_{r0}[(n_2 + 1)\Pi_{r-i,n_2+1} - (a_1\delta_{ik_1} + a_2)\Pi_{r-i,n_2}] \\ - a_1 \sum_{r=i-1}^{k_1} p_{r0} \Pi_{r-i+1,n_2} = 0, \quad (123)$$

for $n_1 + 2 \leq i \leq k_1$.

The boundary condition corresponding to $j = n_2$ in (119) may be reduced, with the help of (10), to that in (121) for $j = n_2$. However, this last condition may be shown to be redundant, and it suffices to use (121) for $0 \leq j \leq n_2 - 1$. These conditions, together with those in (122) and (123), determine the constants D_m , $m = 0, \dots, n_2$ and the probabilities p_{r0} , $r = n_1 + 1, \dots, k_1$, only to within a multiplicative constant, which is determined from the normalization condition (110). But, from (111), with the help of (10) and (171), it follows that

$$\sum_{j=0}^{n_2} p_{ij} = D_0 s_i(0, a_1)s_{n_2}(1, a_2), \quad 0 \leq i \leq n_1. \quad (124)$$

Also, from (109), if we sum on j from 0 to n_2 , and on i from ℓ to k_1 , we

obtain

$$n_1 \sum_{j=0}^{n_2} p_{\ell j} = a_1 p_{\ell-1, n_2}, \quad n_1 + 1 \leq \ell \leq k_1. \quad (125)$$

Hence, from (124) and (125), with the help of (111), (118), and (171), the normalization condition (110) implies that

$$D_0 s_{n_1}(1, a_1) s_{n_2}(1, a_2) + \frac{a_1}{n_1} \sum_{m=0}^{n_2} D_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) + \frac{a_1}{n_1} \left(\sum_{r=n_1+1}^{k_1} p_{r0} \sum_{\ell=0}^{r-n_1-1} \Pi_{\ell, n_2} - p_{k_1, 0} \Pi_{0, n_2} \right) = 1. \quad (126)$$

Once the constants D_m , $m = 0, \dots, n_2$ and the probabilities p_{r0} , $r = n_1 + 1, \dots, k_1$ have been determined, the remaining probabilities may be calculated from (111) and (118). Also, other steady-state quantities of interest may be calculated directly. In particular, the loss probabilities are

$$L_1 = p_{k_1, n_2} = p_{k_1, 0} \Pi_{0, n_2}, \quad (127)$$

and, with the help of (171),

$$L_2 = \sum_{i=0}^{k_1} p_{i, n_2} = \sum_{m=0}^{n_2} D_m s_{n_1}(1 + \rho_m, a_1) s_{n_2}(-\rho_m, a_2) + \sum_{r=n_1+1}^{k_1} p_{r0} \sum_{\ell=0}^{r-n_1-1} \Pi_{\ell, n_2}. \quad (128)$$

The probability that a demand from the primary source is queued on arrival is

$$Q_1 = \sum_{i=n_1}^{k_1-1} p_{i, n_2} = \sum_{m=0}^{n_2} D_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) + \sum_{r=n_1+1}^{k_1} p_{r0} \sum_{\ell=0}^{r-n_1-1} \Pi_{\ell, n_2} - p_{k_1, 0} \Pi_{0, n_2}, \quad (129)$$

which may be simplified in view of (126).

The probability that a demand arriving from the primary source overflows (immediately) is

$$I_{12} = \sum_{i=n_1}^{k_1} \sum_{j=0}^{n_2-1} p_{ij} = \sum_{j=0}^{n_2-1} \left(p_{n_1, j} + \sum_{i=n_1+1}^{k_1} p_{ij} \right). \quad (130)$$

With the help of (125) and (171), it is found that

$$I_{12} = \sum_{m=0}^{n_2} D_m s_{n_1}(\rho_m, a_1) \left[s_{n_2-1}(1 - \rho_m, a_2) + \frac{a_1}{n_1} s_{n_2}(-\rho_m, a_2) \right] + \left(\frac{a_1}{n_1} - 1 \right) \sum_{r=n_1+1}^{k_1} p_{r0} \sum_{\ell=0}^{r-n_1-1} \Pi_{\ell, n_2} - \frac{a_1}{n_1} p_{k_1, 0} \Pi_{0, n_2}. \quad (131)$$

The mean departure rate from the primary queue to the primary servers is

$$R_{11} = n_1 \mu \sum_{i=n_1+1}^{k_1} \sum_{j=0}^{n_2} p_{ij}. \quad (132)$$

It follows from (1) and (125) that $R_{11} = \lambda_1 Q_1$, as is to be expected.

The average number of demands in the primary queue is

$$V_1 = \sum_{i=n_1+1}^{k_1} \sum_{j=0}^{n_2} (i - n_1) p_{ij}. \quad (133)$$

Again with the help of (125), it is found that

$$V_1 = \frac{a_1}{n_1} \left[\sum_{m=0}^{n_2} D_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) - (q_1 + 1) p_{k_1,0} \Pi_{0,n_2} \right. \\ \left. + \sum_{r=n_1+1}^{k_1} p_{r0} \sum_{\ell=0}^{r-n_1-1} (r + 1 - \ell - n_1) \Pi_{\ell,n_2} \right]. \quad (134)$$

The average number of demands in service in the primary is

$$X_1 = \sum_{i=0}^{k_1} \sum_{j=0}^{n_2} \min(i, n_1) p_{ij}. \quad (135)$$

With the help of (110), (124) and (172), we obtain

$$X_1 = n_1 - D_0 s_{n_1-1}(2, a_1) s_{n_2}(1, a_2). \quad (136)$$

The average waiting time of the demands which are queued in the primary is W_1 , as given by (87). Also, since $R_{12} = 0$ and $\lambda_1 = a_1 \mu$, the first relationship in (88) becomes

$$a_1(1 - L_1 - I_{12}) = X_1. \quad (137)$$

In view of (127), (131), and (136), this relationship provides a useful numerical check. Finally, we comment that the only Π_{rj} which occur in (127) to (129), (131), and (134) are Π_{ℓ,n_2} , for $\ell = 0, \dots, q_1 - 1$.

VIII. PRIMARY AND SECONDARY QUEUES: CASE II

We now consider the second case when there is a secondary queue, with $q_2 \geq 1$ waiting spaces, as well as a primary queue, with $q_1 \geq 1$ waiting spaces. The steady-state probabilities satisfy the equations

$$[a_1(1 - \delta_{ik_1} \chi_{j-n_2}) + a_2(1 - \delta_{jk_2}) + \min(i, n_1) + \min(j, n_2)] p_{ij} \\ = a_1(1 - \delta_{i0})(1 - \chi_{i-n_1-1} \chi_{n_2-1-j}) p_{i-1,j} \\ + (1 - \delta_{j0})(a_1 \chi_{i-n_1} \chi_{n_2-j} + a_2) p_{i,j-1} + (1 - \delta_{ik_1}) \min(i + 1, n_1) p_{i+1,j} \\ + (1 - \delta_{jk_2}) \min(j + 1, n_2) p_{i,j+1}, \quad (138)$$

for $0 \leq i \leq k_1$, $0 \leq j \leq k_2$, where k_1 and k_2 are as defined in (59). The

normalization condition is

$$\sum_{i=0}^{k_1} \sum_{j=0}^{k_2} p_{ij} = 1. \quad (139)$$

Now (138) for $0 \leq i \leq n_1 - 1$ and $0 \leq j \leq k_2$ leads, analogously to case I, to the representation

$$p_{ij} = \begin{cases} \sum_{m=0}^{k_2} C_m s_i(\rho_m, a_1) s_j(-\rho_m, a_2) \theta_{n_1}(\rho_m) \phi_{n_2}(\rho_m), & 0 \leq j \leq n_2, \\ \sum_{m=0}^{k_2} C_m s_i(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \theta_{n_1}(\rho_m) \phi_j(\rho_m), & n_2 \leq j \leq k_2, \end{cases} \quad (140)$$

for $0 \leq i \leq n_1$, where ρ_m , $m = 0, \dots, k_2$ are the roots of (30), with $\rho_0 = 0$, and the constants C_m are to be determined. Similarly, (138) for $n_1 + 1 \leq i \leq k_1$ and $n_2 + 1 \leq j \leq k_2$, analogously to case I, leads to the representation

$$p_{ij} = \sum_{m=0}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \theta_i(\rho_m) \phi_j(\rho_m) + \sum_{\ell=1}^{q_1} B_\ell \theta_i(\nu_\ell) \phi_j(\nu_\ell), \quad n_1 + 1 \leq i \leq k_1, \quad n_2 + 1 \leq j \leq k_2, \quad (141)$$

where ν_ℓ , $\ell = 1, \dots, q_1$, are the roots of $\theta_{n_1}(\nu) = 0$, and the constants B_ℓ are to be determined. As before, the representations in (140) and (141) agree for $i = n_1$.

Finally, (138) for $n_1 + 1 \leq i \leq k_1$ and $0 \leq j \leq n_2 - 1$ leads, as in the previous section, to the representation

$$p_{ij} = \sum_{r=i}^{k_1} p_{r0} \Pi_{r-i, j}, \quad n_1 + 1 \leq i \leq k_1, \quad 0 \leq j \leq n_2. \quad (142)$$

In order that the representations in (141) and (142) agree for $j = n_2$, we must have

$$\sum_{r=i}^{k_1} p_{r0} \Pi_{r-i, n_2} = \sum_{m=0}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \theta_i(\rho_m) \phi_{n_2}(\rho_m) + \sum_{\ell=1}^{q_1} B_\ell \theta_i(\nu_\ell) \phi_{n_2}(\nu_\ell), \quad n_1 + 1 \leq i \leq k_1. \quad (143)$$

Since, from (116) and (167), $\Pi_{0, n_2} = s_{n_2}(n_1, a_1 + a_2) > 0$, these equations may be used to solve successively for $p_{k_1, 0}, \dots, p_{n_1+1, 0}$ in terms of C_m , $m = 0, \dots, k_2$ and B_ℓ , $\ell = 1, \dots, q_1$.

It remains to satisfy the boundary conditions at $i = n_1$, $0 \leq j \leq k_2$ and at $j = n_2$, $n_1 + 1 \leq i \leq k_1$, as well as the normalization condition (139). Hence, from (138), in addition to (119) for $0 \leq j \leq n_2 - 1$ and

(71) for $n_2 + 1 \leq j \leq k_2$, we must have

$$(a_1 + a_2 + n_1 + n_2)p_{n_1, n_2} = a_1 p_{n_1-1, n_2} + (a_1 + a_2)p_{n_1, n_2-1} + n_1 p_{n_1+1, n_2} + n_2 p_{n_1, n_2+1}, \quad (144)$$

and

$$[a_1(1 - \delta_{ik_1}) + a_2 + n_1 + n_2]p_{i, n_2} = a_1 p_{i-1, n_2} + (a_1 + a_2)p_{i, n_2-1} + n_1(1 - \delta_{ik_1})p_{i+1, n_2} + n_2 p_{i, n_2+1}, \quad (145)$$

for $n_1 + 1 \leq i \leq k_1$.

From (119) for $0 \leq j \leq n_2 - 1$, (140) and (142), it is found, analogously to (121), that

$$\sum_{m=0}^{k_2} C_m [\rho_m s_{n_1}(1 + \rho_m, a_1) s_j(-\rho_m, a_2) + a_1 s_{n_1}(\rho_m, a_1) s_j(-1 - \rho_m, a_2)] \theta_{n_1}(\rho_m) \phi_{n_2}(\rho_m) - n_1 \sum_{r=n_1+1}^{k_1} p_{r0} \Pi_{r-n_1-1, j} = 0, \quad (146)$$

for $0 \leq j \leq n_2 - 1$. Also, from (71), (140), and (141), it is found, analogously to (74), that

$$\sum_{m=0}^{k_2} C_m \rho_m s_{n_2}(-\rho_m, a_2) [s_{n_1}(1 + \rho_m, a_1) \theta_{n_1}(\rho_m) + s_{n_1}(\rho_m, a_1) \Omega_{q_1-1}(\rho_m)] \phi_j(\rho_m) = n_1 \sum_{\ell=1}^{q_1} B_\ell \theta_{n_1+1}(\nu_\ell) \phi_j(\nu_\ell), \quad (147)$$

for $n_2 + 1 \leq j \leq k_2$. From (141), (142), and (145), with the help of (177) and (180), it follows that

$$a_2 \sum_{m=0}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \theta_i(\rho_m) \phi_{n_2-1}(\rho_m) + a_2 \sum_{\ell=1}^{q_1} B_\ell \theta_i(\nu_\ell) \phi_{n_2-1}(\nu_\ell) - (a_1 + a_2) \sum_{r=i}^{k_1} p_{r0} \Pi_{r-i, n_2-1} = 0, \quad (148)$$

for $n_1 + 1 \leq i \leq k_1$.

The boundary condition (144), which may be reduced with the help of (30) to that in (146) for $j = n_2$, can be shown to be redundant. The conditions in (143) and (146) to (148) determine the constants C_m , $m = 0, \dots, k_2$ and B_ℓ , $\ell = 1, \dots, q_1$, and the probabilities p_{r0} , $r = n_1 + 1, \dots, k_1$, only to within a multiplicative constant, which is determined from the normalization condition (139). But, from (140), with the help

of (27), (30), and (171), it follows that

$$\sum_{j=0}^{k_2} p_{ij} = C_0 s_i(0, a_1) \theta_{n_1}(0) [s_{n_2}(1, a_2) \Psi_{q_2}(0) - s_{n_2-1}(1, a_2) \Psi_{q_2-1}(0)], \quad (149)$$

for $0 \leq i \leq n_1$. Also, from (138), if we sum on j from 0 to k_2 , and on i from ℓ to k_1 , we obtain

$$n_1 \sum_{j=0}^{k_2} p_{\ell j} = a_1 \sum_{j=n_2}^{k_2} p_{\ell-1, j}, \quad n_1 + 1 \leq \ell \leq k_1. \quad (150)$$

Hence, from (149) and (150), with the help of (27), (66), (141), and (171), the normalization condition (139) implies that

$$\begin{aligned} C_0 s_{n_1}(1, a_1) \theta_{n_1}(0) [s_{n_2}(1, a_2) \Psi_{q_2}(0) - s_{n_2-1}(1, a_2) \Psi_{q_2-1}(0)] \\ + \frac{a_1}{n_1} \sum_{m=0}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) [\Omega_{q_1}(\rho_m) - 1] \Psi_{q_2}(\rho_m) \\ + \frac{a_1}{n_1} \sum_{\ell=1}^{q_1} B_{\ell} [\Omega_{q_1}(\nu_{\ell}) - 1] \Psi_{q_2}(\nu_{\ell}) = 1. \end{aligned} \quad (151)$$

Once the constants C_m , $m = 0, \dots, k_2$ and B_{ℓ} , $\ell = 1, \dots, q_1$, and the probabilities p_{r0} , $r = n_1 + 1, \dots, k_1$ have been determined, the remaining probabilities may be calculated from (140) to (142).

IX. SOME STEADY-STATE QUANTITIES: CASE II

Other steady-state quantities of interest are depicted in the diagram of Fig. 1. The loss probabilities L_1 and L_2 are given by (78), and the queuing probabilities Q_1 and Q_2 are given by (79). Also, the mean departure rate from the secondary queue, R_{22} is given by (82). The average number of demands in the secondary queue, V_2 , is as given by (84), but the average number of demands in the primary queue is

$$V_1 = \sum_{i=n_1+1}^{k_1} \sum_{j=0}^{k_2} (i - n_1) p_{ij}. \quad (152)$$

The average waiting times of the demands which are queued in the primary or in the secondary, W_1 and W_2 , are given by (87). The (immediate) overflow probability I_{12} is given by (130). The mean departure rate from the primary queue to the primary servers is

$$R_{11} = n_1 \mu \sum_{i=n_1+1}^{k_1} \sum_{j=0}^{k_2} p_{ij}, \quad (153)$$

and the average number of demands in service in the primary is

$$X_1 = \sum_{i=0}^{k_1} \sum_{j=0}^{k_2} \min(i, n_1) p_{ij}. \quad (154)$$

The above steady-state quantities may be expressed in terms of the

constants C_m and B_ℓ , with the help of the representations in (140) and (141) and the relationship (150). Because of (150), we do not have to use the representation in (142), and hence we are able to eliminate the quantities $\Pi_{\ell j}$. From (78), analogously to case I, and corresponding to (89) and (90),

$$L_1 = \sum_{m=0}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \Psi_{q_2}(\rho_m) + \sum_{\ell=1}^{q_1} B_\ell \Psi_{q_2}(\nu_\ell) \quad (155)$$

and

$$L_2 = \sum_{m=0}^{k_2} C_m s_{n_2}(-\rho_m, a_2) [s_{n_1}(1 + \rho_m, a_1) \theta_{n_1}(\rho_m) + s_{n_1}(\rho_m, a_1) \Omega_{q_1-1}(\rho_m)] + \sum_{\ell=1}^{q_1} B_\ell \Omega_{q_1-1}(\nu_\ell). \quad (156)$$

Similarly, from (79) we obtain, corresponding to (91),

$$Q_1 = \sum_{m=0}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) [\Omega_{q_1}(\rho_m) - 1] \Psi_{q_2}(\rho_m) + \sum_{\ell=1}^{q_1} B_\ell [\Omega_{q_1}(\nu_\ell) - 1] \Psi_{q_2}(\nu_\ell), \quad (157)$$

which may be simplified in view of (151).

From (130), with the help of (27), (66), (150), and (171) it is found that

$$\begin{aligned} I_{12} = & \sum_{m=0}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2-1}(1 - \rho_m, a_2) \theta_{n_1}(\rho_m) \phi_{n_2}(\rho_m) \\ & + \sum_{m=0}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \\ & \cdot \left\{ \frac{a_1}{n_1} [\Omega_{q_1}(\rho_m) - 1] - \Omega_{q_1-1}(\rho_m) \right\} \Psi_{q_2}(\rho_m) \\ & + \sum_{\ell=1}^{q_1} B_\ell \left\{ \frac{a_1}{n_1} [\Omega_{q_1}(\nu_\ell) - 1] - \Omega_{q_1-1}(\nu_\ell) \right\} \Psi_{q_2}(\nu_\ell). \end{aligned} \quad (158)$$

From (1), (150), and (153), it follows directly that $R_{11} = \lambda_1 Q_1$, as is to be expected. From (154), with the help of (139), (149), and (172), it follows that

$$X_1 = n_1 - C_0 s_{n_1-1}(2, a_1) \theta_{n_1}(0) [s_{n_2}(1, a_2) \Psi_{q_2}(0) - s_{n_2-1}(1, a_2) \Psi_{q_2-1}(0)]. \quad (159)$$

In view of (155), (158), and (159), the relationship in (137) provides a

useful numerical check. From (152), with the help of (27), (51), (66), (150), (182), and (183), it is found that

$$\begin{aligned}
 V_1 = & C_0 s_{n_1}(0, a_1) s_{n_2}(0, a_2) \Psi_{q_2}(0) \Delta_{q_1} \left(\frac{n_1}{a_1} \right) \\
 & + \frac{a_1}{n_1} \sum_{m=1}^{k_2} C_m s_{n_1}(\rho_m, a_1) s_{n_2}(-\rho_m, a_2) \Psi_{q_2}(\rho_m) \\
 & \cdot \left\{ \Omega_{q_1}(\rho_m) - (q_1 + 1) - \frac{a_1}{\rho_m} [1 - \Omega_{q_1}(\rho_m) + \frac{n_1}{a_1} \Omega_{q_1-1}(\rho_m)] \right\} \\
 & + \frac{a_1}{n_1} \sum_{\ell=1}^{q_1} B_\ell \Psi_{q_2}(\nu_\ell) \left\{ \Omega_{q_1}(\nu_\ell) - (q_1 + 1) \right. \\
 & \quad \left. - \frac{a_1}{\nu_\ell} [1 - \Omega_{q_1}(\nu_\ell) + \frac{n_1}{a_1} \Omega_{q_1-1}(\nu_\ell)] \right\}. \quad (160)
 \end{aligned}$$

Now (60) and (138) are identical for $0 \leq i \leq k_1$, $n_2 + 1 \leq j \leq k_2$. Hence, if we define r_j as in (98), and proceed as in case I, we deduce that (50) and (52) hold, and $R_{22} = \lambda_2 Q_2$, as is to be expected.

Although we assumed here that $q_2 \geq 1$, we remark that the expressions in (155) to (160) remain valid for $q_2 = 0$. This may be verified from (127) to (129), (131), (134), and (136), by substituting $D_m = C_m \theta_{n_1}(\rho_m)$ and using (143), noting that $\phi_{k_2}(\rho) \equiv 1$, and $k_2 = n_2$ for $q_2 = 0$. We point out that

$$\sum_{r=n_1+1}^{k_1} p_{r0} \sum_{\ell=0}^{r-n_1-1} \Pi_{\ell, n_2} = \sum_{i=n_1+1}^{k_1} \sum_{r=i}^{k_1} p_{r0} \Pi_{r-i, n_2}, \quad (161)$$

and

$$\begin{aligned}
 \sum_{r=n_1+1}^{k_1} p_{r0} \sum_{\ell=0}^{r-n_1-1} (r+1-\ell-n_1) \Pi_{\ell, n_2} \\
 = \sum_{i=n_1+1}^{k_1} (i+1-n_1) \sum_{r=i}^{k_1} p_{r0} \Pi_{r-i, n_2}. \quad (162)
 \end{aligned}$$

Similarly, the normalization condition (151) holds for $q_2 = 0$, as may be verified from (126). Also, the boundary condition (146) reduces to (121) for $q_2 = 0$. Finally, it may be verified, with the help of (113) and (143), that the boundary conditions (122) and (123) are equivalent to (148) for $q_2 = 0$.

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APPENDIX A

We define $s_i(\lambda, a)$ by the recurrence relation

$$(a + i + \lambda)s_i(\lambda, a) = a(1 - \delta_{i0})s_{i-1}(\lambda, a) + (i + 1)s_{i+1}(\lambda, a);$$

$$s_0(\lambda, a) = 1, \quad (163)$$

for $i = 0, 1, \dots$. Thus $s_n(\lambda, a)$ is a polynomial of degree n in both λ and a , and it may be related to a Poisson-Charlier polynomial.^{7,8} However, we give here the properties of $s_n(\lambda, a)$ which we will need. It follows from (163) that the generating function

$$g(u) = \sum_{i=0}^{\infty} s_i(\lambda, a)u^i \quad (164)$$

satisfies the equation

$$(1 - u) \frac{dg}{du} = [a(1 - u) + \lambda]g; \quad g(0) = 1. \quad (165)$$

Hence,

$$\sum_{i=0}^{\infty} s_i(\lambda, a)u^i = e^{au}(1 - u)^{-\lambda}. \quad (166)$$

This leads to the explicit formula

$$s_i(\lambda, a) = \sum_{r=0}^i \frac{(\lambda)_r a^{i-r}}{r!(i-r)!}, \quad (167)$$

where

$$(\lambda)_0 = 1, \quad (\lambda)_r = \lambda(\lambda + 1) \dots (\lambda + r - 1), \quad r = 1, 2, \dots \quad (168)$$

If we differentiate (166) with respect to u , we obtain

$$(i + 1)s_{i+1}(\lambda, a) = as_i(\lambda, a) + \lambda s_i(\lambda + 1, a). \quad (169)$$

Also, if we write $(1 - u)^{-\lambda} = (1 - u)(1 - u)^{-\lambda-1}$, it follows from (166) that

$$s_i(\lambda, a) = s_i(\lambda + 1, a) - (1 - \delta_{i0})s_{i-1}(\lambda + 1, a). \quad (170)$$

From (170), we obtain

$$\sum_{i=0}^n s_i(\lambda, a) = s_n(\lambda + 1, a). \quad (171)$$

From (169) and (171), we deduce that

$$\sum_{i=0}^n (n - i)s_i(\lambda, a) = (1 - \delta_{n0})s_{n-1}(\lambda + 2, a). \quad (172)$$

We now turn our attention to the Chebyshev polynomials of the second kind,⁹ $U_\ell(x)$. They may be defined by the recurrence relation

$$2x U_\ell(x) = U_{\ell+1}(x) + U_{\ell-1}(x); \quad U_{-1}(x) \equiv 0, U_0(x) \equiv 1, \quad (173)$$

for $\ell = 0, 1, \dots$. From (26) and (173), it follows that

$$(a_2 + n_2 - \rho) \Psi_\ell(\rho) = a_2 \Psi_{\ell+1}(\rho) + n_2 \Psi_{\ell-1}(\rho);$$

$$\Psi_{-1}(\rho) \equiv 0, \Psi_0(\rho) \equiv 1. \quad (174)$$

Since⁹

$$U_\ell \left[\frac{1}{2} \left(z + \frac{1}{z} \right) \right] = \sum_{r=0}^{\ell} z^{2r-\ell}, \quad (175)$$

it also follows that

$$\Psi_\ell(0) = \left(\frac{n_2}{a_2} \right)^{\ell/2} U_\ell \left(\frac{a_2 + n_2}{2\sqrt{a_2 n_2}} \right) = \sum_{r=0}^{\ell} \left(\frac{n_2}{a_2} \right)^r. \quad (176)$$

From (27) and (174), we deduce that

$$[a_2(1 - \delta_{jk_2}) + n_2 - \rho] \phi_j(\rho) = a_2 \phi_{j-1}(\rho) + n_2(1 - \delta_{jk_2}) \phi_{j+1}(\rho), \quad (177)$$

for $j \leq k_2$.

Similarly, from (65) and (173), it follows that

$$(a_1 + n_1 + \nu) \Omega_\ell(\nu) = a_1 \Omega_{\ell+1}(\nu) + n_1 \Omega_{\ell-1}(\nu);$$

$$\Omega_{-1}(\nu) \equiv 0, \Omega_0(\nu) \equiv 1. \quad (178)$$

Also, from (175), we obtain

$$\Omega_\ell(0) = \left(\frac{n_1}{a_1} \right)^{\ell/2} U_\ell \left(\frac{a_1 + n_1}{2\sqrt{a_1 n_1}} \right) = \sum_{r=0}^{\ell} \left(\frac{n_1}{a_1} \right)^r. \quad (179)$$

From (66) and (178), we deduce that

$$[a_1(1 - \delta_{ik_1}) + n_1 + \nu] \theta_i(\nu) = a_1 \theta_{i-1}(\nu) + n_1(1 - \delta_{ik_1}) \theta_{i+1}(\nu), \quad (180)$$

for $i \leq k_1$. Also, from (179),

$$\theta_i(0) = \left(\frac{n_1}{a_1} \right)^{k_1-i}. \quad (181)$$

Hence, since $k_1 = n_1 + q_1$,

$$\sum_{i=n_1+1}^{k_1} (i - n_1) \theta_i(0) = \Delta_{q_1} \left(\frac{n_1}{a_1} \right), \quad (182)$$

where $\Delta_q(\xi)$ is as defined in (51). If we multiply (180) by $(i - n_1)$ and sum on i , we obtain

$$\begin{aligned} \nu \sum_{i=n_1+1}^{k_1} (i - n_1) \theta_i(\nu) &= a_1 \sum_{i=n_1}^{k_1-1} \theta_i(\nu) - n_1 \sum_{i=n_1+1}^{k_1} \theta_i(\nu) \\ &= a_1 [\Omega_{q_1}(\nu) - 1] - n_1 \Omega_{q_1-1}(\nu), \end{aligned} \quad (183)$$

from (66). We also note, from (178), that

$$a_1 \theta_{n_1}(\nu) - n_1 \theta_{n_1+1}(\nu) = \nu \Omega_{q_1-1}(\nu). \quad (184)$$

APPENDIX B

We first consider the roots of $s_n(1 - \rho, a) = 0$, where $a > 0$. We relate this to an eigenvalue problem for a real symmetric matrix, so that we may apply a result of Slepian and Landau¹² to prove that the roots are distinct. For $j = 0, 1, \dots$, we let

$$y_j = \sqrt{\frac{j!}{a^j}} s_j(1 - \rho, a). \quad (185)$$

Then, from (163) it follows that

$$\sqrt{j} y_{j-1} + \frac{1}{\sqrt{a}} (\rho - 1 - a - j) y_j + \sqrt{j+1} y_{j+1} = 0. \quad (186)$$

If $s_n(1 - \rho, a) = 0$, then $y_n = 0$ and

$$\left[\mathbf{A}_n - \frac{1}{\sqrt{a}} (1 + a - \rho) \mathbf{I}_n \right] (y_0, \dots, y_{n-1})^T = 0, \quad (187)$$

where T denotes transpose, \mathbf{I}_n is the $n \times n$ unit matrix, and

$$\mathbf{A}_1 = [0], \quad \mathbf{A}_n = \begin{bmatrix} 0 & 1 & & \\ 1 & \frac{-1}{\sqrt{a}} & \sqrt{2} & \\ & \text{---} & \text{---} & \text{---} \\ & & \sqrt{n-1} & -\frac{(n-1)}{\sqrt{a}} \end{bmatrix}, \quad n \geq 2. \quad (188)$$

Hence the equation $s_n(1 - \rho, a) = 0$ is equivalent to

$$\det \left[\mathbf{A}_n - \frac{1}{\sqrt{a}} (1 + a - \rho) \mathbf{I}_n \right] = 0. \quad (189)$$

We remark that Kaufman,⁶ who has analyzed the same overflow problem using matrix techniques, arrived at the same eigenvalue problem by block diagonalization.

Now \mathbf{A}_n is a real symmetric matrix, so that its eigenvalues are real.

Moreover, \mathbf{A}_n may be obtained from \mathbf{A}_{n+1} by deleting the last row and column. Hence we may apply the results of Slepian and Landau.¹² In order to establish that the eigenvalues of \mathbf{A}_n separate those of \mathbf{A}_{n+1} for $n = 1, 2, \dots$, we have to verify that certain quantities (see (6) of Ref. 12) are different from zero. Specifically, let α be the n -dimensional column vector obtained from the last column of \mathbf{A}_{n+1} by omitting the diagonal element, so that

$$\alpha = \begin{cases} (1), & n = 1, \\ (0, \dots, 0, \sqrt{n})^T, & n \geq 2. \end{cases} \quad (190)$$

Hence

$$(\alpha, (y_0, \dots, y_{n-1})^T) = \sqrt{n} y_{n-1} \neq 0, \quad (191)$$

since $y_{n-1} \neq 0$ for any eigenvector corresponding to $y_n = 0$. This establishes the desired result, and we may conclude that the zeros of $s_n(1 - \rho, a)$ separate those of $s_{n+1}(1 - \rho, a)$, for $n = 1, 2, \dots$. In particular, the zeros of $s_n(1 - \rho, a)$ are real and distinct. Since, from (167), $s_n(\lambda, a) > 0$ for $\lambda \geq 0$, it follows that the zeros of $s_n(1 - \rho, a)$ are larger than 1, and hence positive.

We next consider the zeros of

$$P_{n,q}(\rho) \equiv s_n(1 - \rho, a)\psi_q(\rho) - s_{n-1}(1 - \rho, a)\psi_{q-1}(\rho), \quad (192)$$

where

$$\psi_\ell(\rho) = \left(\frac{n}{a}\right)^{\ell/2} U_\ell\left(\frac{a + n - \rho}{2\sqrt{an}}\right). \quad (193)$$

We let

$$z_j = \begin{cases} y_j, & j = 0, \dots, n, \\ z_{n+q-1} U_{n+q-j-1} \left(\frac{a + n - \rho}{2\sqrt{an}}\right), & j = n-1, \dots, n+q, \end{cases} \quad (194)$$

where y_j is as in (185). We note that $z_{n+q} = 0$, since $U_{-1}(x) \equiv 0$. Also, from (185), (192), and (193), the consistency of the definitions of z_{n-1} , and of z_n , in (194) implies that $P_{n,q}(\rho) = 0$. From (186), it follows that

$$\sqrt{j} z_{j-1} + \frac{1}{\sqrt{a}} (\rho - 1 - a - j) z_j + \sqrt{j+1} z_{j+1} = 0, \quad j = 0, \dots, n-1, \quad (195)$$

and from (173) we obtain

$$\sqrt{n} z_{j-1} + \frac{1}{\sqrt{a}} (\rho - a - n) z_j + \sqrt{n} z_{j+1} = 0, \quad j = n, \dots, n+q-1. \quad (196)$$

But $P_{n,0}(\rho) = s_n(1 - \rho, a)$ and we have shown that its zeros are real and distinct. Proceeding as before, it follows that the zeros of $P_{n,q}(\rho)$ separate those of $P_{n,q+1}(\rho)$, for $q = 0, 1, \dots$. In particular, the zeros of $P_{n,q}(\rho)$ are real and distinct.

We now consider $\rho \leq 0$, and will show that $P_{n,q}(\rho) \neq 0$, for $n = 1, 2, \dots$ and $q = 1, 2, \dots$. Since, from (170),

$$s_n(1 - \rho, a) = s_n(-\rho, a) + s_{n-1}(1 - \rho, a), \quad (197)$$

and, from (167), $s_n(\lambda, a) > 0$ for $\lambda \geq 0$, it follows that $s_n(1 - \rho, a) > s_{n-1}(1 - \rho, a) > 0$ for $\rho \leq 0$. Hence

$$\frac{s_n(1 - \rho, a)}{s_{n-1}(1 - \rho, a)} > 1, \quad \rho \leq 0. \quad (198)$$

Also, for $\rho \leq 0$,

$$\frac{a + n - \rho}{2\sqrt{an}} = \frac{1}{2} \left(\eta + \frac{1}{\eta} \right), \quad \eta \geq \max \left(\sqrt{\frac{n}{a}}, \sqrt{\frac{a}{n}} \right). \quad (199)$$

But, from (175),

$$\frac{\eta U_{q-1}[(1/2)(\eta + (1/\eta))]}{U_q[(1/2)(\eta + (1/\eta))]} = \frac{\sum_{r=1}^q \eta^{2r-q}}{\sum_{r=0}^q \eta^{2r-q}} < 1. \quad (200)$$

It follows from (193), (199), and (200), that

$$\frac{\psi_{q-1}(\rho)}{\psi_q(\rho)} < \frac{1}{\eta} \sqrt{\frac{a}{n}} \leq 1, \quad \rho \leq 0. \quad (201)$$

From (192), (198), and (201), we deduce that $P_{n,q}(\rho) \neq 0$ for $\rho \leq 0$. Thus the zeros of $P_{n,q}(\rho)$ are positive.

Finally we consider the zeros of $\theta_{n_1}(\nu)$. For $\ell = 0, 1, 2, \dots$, we define

$$w_\ell(\nu) \equiv U_\ell \left(\frac{a_1 + n_1 + \nu}{2\sqrt{a_1 n_1}} \right) - \sqrt{\frac{a_1}{n_1}} U_{\ell-1} \left(\frac{a_1 + n_1 + \nu}{2\sqrt{a_1 n_1}} \right). \quad (202)$$

Then, from (59), (65), and (66), $\theta_{n_1}(\nu) = 0$ is equivalent to $w_{q_1}(\nu) = 0$. From (173) we deduce that

$$(n_1 + \nu)w_0 = \sqrt{a_1 n_1} w_1, \quad (203)$$

and

$$\sqrt{a_1 n_1} w_{\ell-1} - (a_1 + n_1 + \nu)w_\ell + \sqrt{a_1 n_1} w_{\ell+1} = 0, \quad (204)$$

for $\ell = 1, 2, \dots$. Proceeding as before, it follows that the zeros of $w_q(\nu)$ separate those of $w_{q+1}(\nu)$, for $q = 1, 2, \dots$. In particular, the zeros of $w_q(\nu)$ are real and distinct. But, from (193), (201), and (202), it follows that $w_q(\nu) \neq 0$ for $\nu \geq 0$. Hence, the zeros of $w_q(\nu)$ are negative.

REFERENCES

1. E. Brockmeyer, "Det Simple Overflowproblem i Telefontrafikteorien," *Teleteknik*, 5 (1954), pp. 361-374.
2. L. Kosten, "Über Sperrungswahrscheinlichkeiten bei Staffelschaltungen," *Elekt. Nach. Tech.*, 14 (1937), pp. 5-12.
3. L. Kosten, "Behaviour of Overflow Traffic and the Probabilities of Blocking in Simple Gradings," *Proc. Eighth Int'l. Teletraffic Congress*, Melbourne, 1976, pp. 425-1-5.
4. J. H. Rath, "An Approximation for a Queueing System with Two Queues and Overflows," unpublished work.
5. L. Kaufman, J. B. Seery, and J. A. Morrison, "Numerical Results for Some Overflow Problems with Queueing," unpublished work.
6. L. Kaufman, "Matrix Methods for Queueing Problems," unpublished work.
7. A. Erdélyi et al., *Higher Transcendental Functions*, Vol. II, New York: McGraw-Hill, 1953, p. 226.
8. J. Riordan, *Stochastic Service Systems*, New York: John Wiley, 1962.
9. W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and Theorems for the Special Functions of Mathematical Physics*, New York: Springer-Verlag, 1966, p. 256.
10. L. Kleinrock, *Queueing Systems, Volume I: Theory*, New York: John Wiley, 1975.
11. S. Karlin, *A First Course in Stochastic Processes*, New York: Academic Press, 1966.
12. D. Slepian and H. J. Landau, "A Note on the Eigenvalues of Hermitian Matrices," *SIAM J. Math. Anal.*, 9, No. 2 (April 1978), pp. 291-297.
13. U. Herzog, L. Woo, and K. M. Chandy, "Solution of Queueing Problems by a Recursive Technique," *IBM J. Res. Develop.*, 19, No. 3 (May 1975), pp. 295-300.
14. A. Brandwajn, "An Iterative Solution of Two-Dimensional Birth and Death Processes," *Operations Res.*, 27, No. 3 (May-June 1979), pp. 595-605.
15. J. A. Morrison, "Some Traffic Overflow Problems with a Large Secondary Queue," *B.S.T.J.*, this issue, pp. 1463-1482.